Path integral treatment of the hydrogen atom in a curved space of constant curvature. II.

Hyperbolic space

Path integral treatment of the hydrogen atom in a curved space of constant curvature: II. Hyperbolic space

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Abstract. The path integral treatment of the hydrogen atom in a hyperbolic space is discussed. We show by mapping the radial path integral into the SU(1, 1) group manifold that the system has a dynamical SU(1, 1) symmetry. The energy spectrum and normalised energy eigenfunctions are calculated. In the flat-space limit, the standard hydrogen spectrum and corresponding normalised wavefunctions are regained.

1. Introduction

Quantum mechanics in a space of constant negative curvature has attracted considerable attention in recent years. A free particle moving in a compact two-dimensional hyperbolic space is known in classical dynamics to be a chaotic system [1]. The anisotropic Coulomb problem is another example which exhibits a chaotic behaviour [2]. The quantisation of such problems is not a simple task. Usually, the quantum behaviour of such systems have been analysed semiclassically or numerically via Selberg's trace formula [1, 2]. It is certainly interesting to study exactly soluble quantum mechanical systems in a space of constant negative curvature.

The purpose of the present paper is to quantise the hydrogen atom in a hyperbolic space—a space of constant negative curvature—by path integration. Very recently, we have calculated the path integral for the Coulomb problem in a spherical space—a space of constant positive curvature—by utilising the SU(1, 1) dynamical symmetry of the system [3]. Many steps of the present calculation for the negative curvature case are similar to those of the previous one for the positive curvature [3]. However, there are a number of features different from those of the previous case. Avoiding repetition of detailed steps that have been done previously, we wish to explore the new features of the hydrogen atom in a hyperbolic space. While the present article is designated as part II: hyperbolic space, the previous paper [3] should be understood as part I: spherical space.

In section 2, we briefly discuss the geometry of a negatively curved space, embedding it in a four-dimensional Minkowski space. In section 3, we deal with the classical

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dynamics in hyperbolic space. The Coulomb potential and its effect on a classical particle is also briefly discussed. It is a remarkable fact that for large angular momenta the effective radial potential has no minimum point. Consequently, there is an upper limit for the angular momentum of bound states. After the study of classical dynamics, we pursue the path integral quantisation on hyperbolic space. First, we perform the angular integration. Then we apply a local spacetime transformation to change the radial path integral into that of the modified Pöschl–Teller potential [4, 5]. The modified Pöschl–Teller oscillator has been shown to have an SU(1, 1) dynamical symmetry [6] which is now explicitly realised in the path integral by introducing two additional angular variables via the dimensional extension technique [7, 8]. In this fashion, the radial path integral of the hydrogen atom in the hyperbolic space is mapped into that of a free particle moving on the dynamical group manifold of SU(1, 1). Performing the SU(1, 1) path integral we then obtain the energy spectrum and normalised wavefunctions of the Coulomb problem in the negatively curved space. As in the classical case, there is in the quantum case an upper limit for the angular momentum of bound states. In the flat-space limit, the energy spectrum and the normalised wavefunctions of the usual hydrogen atom are obtained.

2. Geometry of a negatively curved space

The hyperbolic space we consider in the present work is a manifold of constant negative curvature $K = -R^{-2} < 0$. The line element in this space is given in polar coordinates as

$$ds^2 = (1 + r^2/R^2)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2)$$

$$0 \le r < \infty \qquad 0 \le \theta < \pi \qquad 0 \le \phi < 2\pi.$$

(2.1)

With $r = R \sinh \chi$, it can also be put in the form

$$ds^2 = R^2 \, d\chi^2 + R^2 \sinh^2 \chi \, (d\theta^2 + \sin^2 \theta \, d\phi^2).$$

(2.2)

In this parametrisation, the metric tensor is

$$g_{ij} = \text{diag}\{R^2, R^2 \sinh^2 \chi, R^2 \sinh^2 \chi \sin^2 \theta\}.$$

(2.3)

The Laplace–Beltrami operator, which is in general defined as $\Delta = g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j)$ with $g = \det |g_{ij}|$ and $g^{ij}g_{jk} = \delta^i_k$, takes the form

$$\Delta = \frac{1}{R^2 \sinh^2 \chi} \frac{\partial}{\partial \chi} \sinh^2 \chi \frac{\partial}{\partial \chi} + \frac{1}{R^2 \sinh^2 \chi \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{R^2 \sinh^2 \chi \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

(2.4)

In the flat-space limit $R \to \infty$, it reduces to the standard Laplacian in polar coordinates.

Spaces of constant curvature in n dimensions can be embedded in a flat space in $n + 1$ dimensions. In the present case we may put the three-dimensional hyperbolic space into a four-dimensional Minkowski space by setting [9]

$$x^0 = R \cosh \chi$$
$$x^1 = R \sinh \chi \sin \theta \sin \phi$$
$$x^2 = R \sinh \chi \sin \theta \cos \phi$$
$$x^3 = R \sinh \chi \cos \theta.$$

(2.5)

Obviously, the hyperbolic space may be identified with the upper sheet of the two-sheeted hyperbola $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = R^2 > 0$ with $x^0 > 0$. The line element of this space is

$$- ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{2.6}$$

Note that since the negativly curved space corresponds to a 'timelike' hyperboloid in Minkowski space, the line element (2.6) is restricted to be 'spacelike', $ds^2 > 0$.

3. The classical Coulomb problem

The Lagrangian for a particle moving under the influence of a scalar potential is

$$\mathcal{L} = (M/2)\dot{s}^2 - V(r). \tag{3.1}$$

For a central potential $V(r) = V(\chi)$, the angular momentum is a first integral of the Euler–Lagrange equations obtained from (3.1). Due to the conservation of angular momentum, the classical particle trajectory lies on a two-dimensional plane. Let the angular momentum vector to be in x^3 direction ($\theta = \pi/2$), i.e. $\mathbf{L} = L\,\mathbf{e}_3$. The Lagrangian then reads

$$\mathcal{L} = (MR^2/2)(\dot{\chi}^2 + \sinh^2\chi\,\dot{\phi}^2) - V(\chi). \tag{3.2}$$

The ϕ equation of motion immediately leads to

$$L = MR^2 \sinh^2\chi\,\dot{\phi} = \text{constant} \tag{3.3}$$

which is the magnitude of the conserved angular momentum. The radial equation of motion is

$$MR\ddot{\chi} = -\frac{\partial}{R\,\partial\chi}\,V_{\text{eff}}(\chi)$$

with an effective potential

$$V_{\text{eff}}(\chi) = \frac{L^2}{2MR^2\sinh^2\chi} + V(\chi). \tag{3.4}$$

The Coulomb potential due to a point charge $Q = -Ze^2$ located at the origin $r = 0$ satisfies the equation,

$$\Delta V(\chi) = -4\pi Q\,\delta(r). \tag{3.5}$$

With (2.4), it is easy to show that (3.5) has the following solution[†]:

$$V(\chi) = -\frac{Ze^2}{R}\,(\coth\chi - 1). \tag{3.6}$$

[†] For $\chi > 0$:

$$\Delta V(\chi) = -(Ze^2/R)\frac{1}{R^2\sinh^2\chi}\frac{\partial}{\partial\chi}\sinh^2\chi\frac{\partial}{\partial\chi}\coth\chi = 0.$$

For $\chi \to 0$:

$$\int \Delta V(\chi)d\mathbf{r} = \oint \left[\frac{\partial}{R\partial\chi}V(\chi)\right]df = -4\pi Q.$$

Note that an additional constant is included in (3.6) so as to satisfy the limiting condition that $V(\chi)$ vanishes as χ tends to infinity.

For the Coulomb problem in a hyperbolic space, the effective potential (3.4) takes the form,

$$V_{\text{eff}}(\chi) = \frac{L^2}{2MR^2 \sinh^2 \chi} - \frac{Ze^2}{R}(\coth \chi - 1) \tag{3.7}$$

or, in dimensionless units,

$$V_{\text{eff}}(\chi)/\kappa = \lambda/\sinh^2 \chi - \coth \chi + 1 \tag{3.8}$$

where

$$\kappa = Ze^2/R \tag{3.9}$$

and $\lambda = L^2/(2MR^2\kappa)$ measures the rotational energy in the scale of the potential energy, κ. As is shown in figure 1, for $\lambda > 1/2$, the effective potential does not have a minimum. Consequently, for large angular momentum the classical Kepler problem does not have bound states. For $\lambda < 1/2$ the minimum of (3.8) is at $\chi_{\text{min}} = \tanh^{-1} 2\lambda$ with $V_{\text{eff}}(\chi_{\text{min}})/\kappa = -(1/2 - \lambda)^2/\lambda$. In figure 1 we display the behaviour of $V_{\text{eff}}(\chi)/\kappa$ for various values of λ. If R is a finite constant, for large χ, the Coulomb potential behaves like $V_{\text{eff}}(\chi)/\kappa \sim 2(2\lambda - 1)e^{-2\chi} \sim (\lambda - 1/2)R^2/r^2$. We expect an upper limit for the angular momentum of bound states in the quantum problem as well.
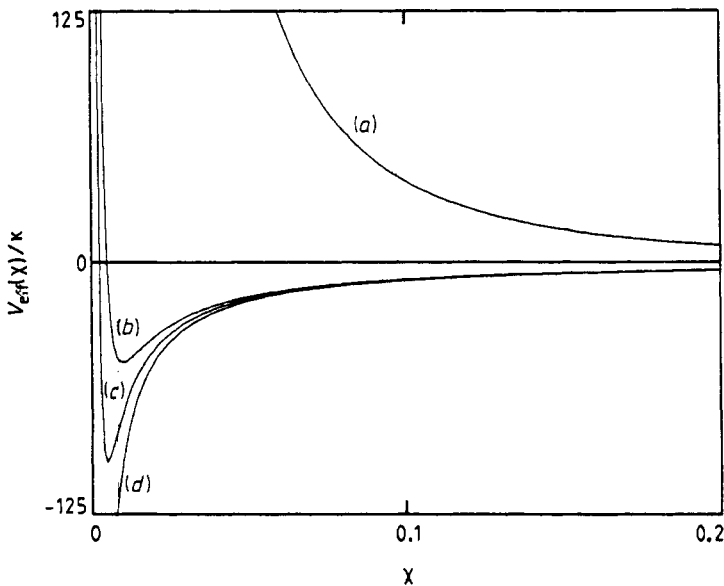
Figure 1. The effective Coulomb potential (3.8) in hyperbolic space for various values of λ. (a) $\lambda = 0.5$, (b) $\lambda = 5.0 \times 10^{-3}$, (c) $\lambda = 2.5 \times 10^{-3}$, (d) $\lambda = 0$.

4. Path integral quantisation in a hyperbolic space

Following our previous path integral approach to the hydrogen atom in spherical space [3] we consider the promotor

$$P(r'', r'; \tau) = \int \exp\left\{ \frac{i}{\hbar} \int (L+E) \, dt \right\} Dr(t) \tag{4.1}$$

from which the energy-dependent Green function $G(r'', r'; E)$ and the propagator $K(r'', r'; t'' - t')$ respectively, can be evaluated by the formulae

$$G(r'', r'; E) = \frac{1}{i\hbar} \int P(r'', r'; \tau) \, d\tau \tag{4.2}$$

$$K(r'', r'; t'' - t') = \frac{1}{2\pi\hbar} \int \int P(r'', r'; \tau) \exp\left\{ -\frac{i}{\hbar} E(t'' - t') \right\} d\tau \, dE. \tag{4.3}$$

In polar coordinates the time sliced version of the path integral (4.1) reads [5]

$$P(r'', r'; \tau) = \lim_{N \to \infty} \int \prod_{j=1}^{N} \left\{ \frac{M}{2\pi i \hbar \tau_j} \right\}^{3/2} \exp\left\{ \frac{i}{\hbar} W_j \right\} \prod_{j=1}^{N-1} R^3 \sinh^2 \chi_j \, d\chi_j \sin \theta_j \, d\theta_j \, d\phi_j \tag{4.4}$$

where $W_j = \int_{t_{j-1}}^{t_j} (L+E) \, dt$ is Hamilton's characteristic function for a short-time interval $\tau_j = t_j - t_{j-1}$. In the present case it is given by

$$W_j = \frac{M}{2\tau_j} (\Delta s_j)^2 + \frac{Ze^2}{R} \tau_j (\coth \chi_j - 1) + E\tau_j. \tag{4.5}$$

We have adopted the standard notation: $r_j = r(t_j)$, $t' = t_0$, $t'' = t_N$.

The metric relation (2.1), which holds only locally, cannot be directly used for a finite time interval. The short-time version of (2.1) for an n-dimensional space of constant negative curvature is [10]

$$(\Delta s_j)^2 = -2R^2(1 - \cosh \omega_j) - n(n-2)\hbar^2 \tau_j^2 / 4M^2 R^2. \tag{4.6}$$

The last term in (4.6) is due to the curvature and is the same as that of an n-dimensional spherical space in magnitude but opposite in sign. In the following we consider only the case $n = 3$. In (4.6) we have also set $\cosh \omega_j = x_{j-1}^\mu x_{\mu j} / R^2$, where x^μ is given by (2.5) and hence

$$\cosh \omega_j = \cosh \chi_{j-1} \cosh \chi_j - \sinh \chi_{j-1} \sinh \chi_j (1 - \cos \Theta_j) \tag{4.7}$$

with $\cos \Theta_j = \cos \theta_j \cos \theta_{j-1} + \sin \theta_j \sin \theta_{j-1} \cos \Delta\phi_j$. The short-time action (4.5) now becomes

$$W_j = \frac{MR^2}{\tau_j} (\cosh \Delta\chi_j - 1) + \frac{MR^2}{\tau_j} \sinh \chi_j \sinh \chi_{j-1} (1 - \cos \Theta_j)$$

$$+ \frac{Ze^2}{R} \tau_j (\coth \chi_j - 1) + \left(E - \frac{3\hbar^2}{8MR^2} \right) \tau_j. \tag{4.8}$$

In the above we realise that the curvature correction in (4.6) amounts to a shift of the energy scale by a constant term. Therefore, we will set $E' = E - 3\hbar^2/8MR^2$. Furthermore, we define, as in section 2, $\kappa = Ze^2/R$. Using the approximation (details can be found in [3]) $(\cosh\Delta\chi_j - 1) \approx \frac{1}{2}(\Delta\chi_j)^2 + (\Delta\chi_j)^4/4!$ and replacing the fourth-order term by an equivalent one, we obtain the following effective short-time action:

$$W_j = \frac{MR^2}{2\tau_j}(\Delta\chi_j)^2 + \frac{MR^2}{\tau_j}\widehat{\sinh^2}\chi_j(1 - \cos\Theta_j) + \kappa\tau_j(\coth\chi_j - 1) + \left(E' - \hbar^2/8MR^2\right)\tau_j \tag{4.9}$$

where $\widehat{\sinh^2}\chi_j = \sinh\chi_j\sinh\chi_{j-1}$. The angular integration has by now become a standard procedure [5, 10, 11] and one immediately finds

$$P(r'', r'; \tau) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} P_l(r'', r'; \tau)\, Y_l^m(\theta'', \phi'')\, Y_l^{m*}(\theta', \phi') \tag{4.10}$$

where the radial promotor is given by

$$P_l(r'', r'; \tau) = (R^3\sinh\chi'\sinh\chi'')^{-1} \lim_{N\to\infty} \int \prod_{j=1}^{N}\left(\frac{MR^2}{2\pi i\hbar\tau_j}\right)^{1/2}\exp\left\{\frac{i}{\hbar}W_j^l\right\}\prod_{j=1}^{N-1}\mathrm{d}\chi_j \tag{4.11}$$

with

$$W_j^l = \frac{MR^2}{2\tau_j}(\Delta\chi_j)^2 - V_{\text{eff}}^q(\chi_j)\tau_j + (E' - \hbar^2/8MR^2)\tau_j \tag{4.12}$$

and

$$V_{\text{eff}}^q(\chi) = \frac{l(l+1)}{2MR^2\sinh^2\chi} - \kappa(\coth\chi - 1) + \frac{\hbar^2}{2MR^2}. \tag{4.13}$$

The effective quantum potential $V_{\text{eff}}^q(\chi)$ of (4.13) with the curvature correction has the minimum at $\chi_{\min} = \tanh^{-1}[2\Lambda l(l+1)]$:

$$V_{\text{eff}}^q(\chi_{\min}) = -\kappa[1/2 - l(l+1)\Lambda]^2/[\Lambda l(l+1)] + \kappa\Lambda \tag{4.14}$$

where $\Lambda = \hbar^2/(2MR^2\kappa)$. As is shown in figure 2, for large χ (i.e. for large r), the potential behaves like

$$V_{\text{eff}}^q(\chi) \simeq -\kappa[1/2 - l(l+1)\Lambda]R^2/r^2 + \kappa\Lambda. \tag{4.15}$$

The bound states occur only when

$$l(l+1)\Lambda < 1/2 \tag{4.16}$$

which sets the upper limit for the angular quantum number l in a way similar to the classical case where $\lambda < 1/2$.

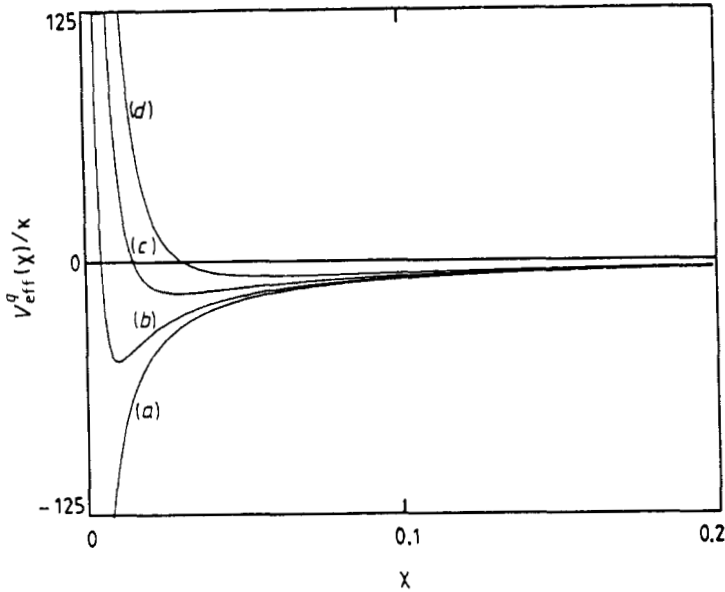The radial path integral will be performed in the next section by utilising the dynamical symmetry of the system.

Figure 2. The effective quantum potential (4.13) in units of κ for $\Lambda = 2.5 \times 10^{-3}$ and various angular momenta. (a) $l = 0$, (b) $l = 1$, (c) $l = 2$, (d) $l = 3$.

5. Realisation of the dynamical SU(1, 1) symmetry

In this section we will perform the integration (4.11) by an explicit realisation of the dynamical SU(1, 1) symmetry of the Coulomb problem in hyperbolic space. We give only the main results as the calculation is similar to that of [3].

We perform the local spacetime transformation $(\chi_j, \tau_j) \to (\beta_j, \sigma_j)$ given by

$$e^{-\beta_j} = \tanh(\chi_j/2) \qquad \sigma_j = -(\tau_j \widehat{\sinh^2 \beta_j})/4 \tag{5.1}$$

with the global time scaling $\sigma = -(\tau \sinh \beta' \sinh \beta'')/4$. Under the transformation (5.1) the kinetic term of the action (4.12) becomes, when terms of $O(\tau^{1+\delta})$ with $\delta > 0$ are neglected,

$$\frac{MR^2}{2\tau_j}(\Delta\chi_j)^2 \simeq \frac{MR^2}{\sigma_j}[1 - \cosh(\Delta\beta_j/2)] + \left[\frac{3}{4} - \frac{1}{\widehat{\sinh^2 \beta_j}}\right]\frac{\hbar^2 \sigma_j}{2MR^2}. \tag{5.2}$$

Similarly the measure changes as

$$(R^3 \sinh \chi' \sinh \chi'')^{-1} \prod_{j=1}^{N}\left(\frac{MR^2}{2\pi i\hbar \tau_j}\right)^{1/2} \prod_{j=1}^{N-1} d\chi_j$$

$$= -R^{-3}(\sinh \beta' \sinh \beta'')^{3/2} \prod_{j=1}^{N}\left(\frac{iMR^2}{8\pi \hbar \sigma_j}\right)^{1/2} \prod_{j=1}^{N-1} d\beta_j. \tag{5.3}$$

From (5.1) also follow $\sinh \chi = 1/\sinh \beta$ and $\coth \chi = \cosh \beta$. Hence the complete short-time action in the new space time variables can be written as

$$W_j = \frac{MR^2}{\sigma_j} \left[1 - \cosh(\Delta \beta_j / 2)\right] + \frac{(2l+1)^2 - 1/4}{2MR^2} \hbar^2 \sigma_j - \left[\frac{E'}{\widehat{\sinh^2}(\beta_j/2)} + \frac{2\kappa - E'}{\widehat{\cosh^2}(\beta_j/2)}\right] \sigma_j. \tag{5.4}$$

This action is formally identical with that of the modified Pöschl–Teller potential having a dynamical SU(1, 1) symmetry [4, 5].
    Setting

$$p = \left[\frac{2MR^2}{\hbar^2}(2\kappa - E) + 1\right]^{1/2} \qquad q = \left[-\frac{2MR^2}{\hbar^2}E + 1\right]^{1/2} \tag{5.5}$$

we use the dimensional extension technique [7, 8] to introduce two additional angular variables ξ and η as in the case of [3]†. Namely, we employ the asymptotic relations valid for small σ_j and integer p and q, respectively:

$$\exp\left\{-\frac{\mathrm{i}(2\kappa - E')\sigma_j}{\hbar \,\widehat{\cosh^2}(\beta_j/2)}\right\} = \left[\frac{MR^2\widehat{\cosh^2}(\beta_j/2)}{2\pi\mathrm{i}\hbar\sigma_j}\right]^{1/2}$$

$$\times \int_0^{2\pi} \exp\left\{\mathrm{i}p\Delta\xi_j + (\mathrm{i}MR^2/\hbar\sigma_j)\widehat{\cosh^2}(\beta_j/2)(1 - \cos\Delta\xi_j)\right\}\mathrm{d}\xi_j \tag{5.6}$$

$$\exp\left\{\frac{-\mathrm{i}E'\sigma_j}{\hbar \,\widehat{\sinh^2}(\beta_j/2)}\right\} = \left[\frac{\mathrm{i}MR^2\widehat{\sinh^2}(\beta_j/2)}{2\pi\hbar\sigma_j}\right]^{1/2}$$

$$\times \int_0^{2\pi} \exp\left\{\mathrm{i}q\Delta\eta_j - (\mathrm{i}MR^2/\hbar\sigma_j)\widehat{\sinh^2}(\beta_j/2)(1 - \cos\Delta\eta_j)\right\}\mathrm{d}\eta_j. \tag{5.7}$$

Finally, we change the variables ξ_j and η_j into Euler angles α_j and γ_j by

$$\alpha_j = \xi_j - \eta_j \qquad \gamma_j = \xi_j + \eta_j \tag{5.8}$$

and

$$\int_0^{2\pi} \mathrm{d}\xi_i \int_0^{2\pi} \mathrm{d}\eta_j = \frac{1}{2} \int_0^{2\pi} \mathrm{d}\alpha_j \int_{-2\pi}^{2\pi} \mathrm{d}\gamma_j. \tag{5.9}$$

For simplicity we set $\alpha' = \gamma' = 0$. Substituting (5.2)–(5.9) into the path integral (4.11) gives:

$$P_l(r'', r'; \tau) = -(2R)^{-3}(\sinh\beta'' \sinh\beta')^2 \exp\left\{\frac{\mathrm{i}\hbar\sigma}{2MR^2}\left[(2l+1)^2 - 1/4\right]\right\}$$

$$\times \int_0^{2\pi} \mathrm{d}\alpha'' \int_{-2\pi}^{2\pi} \mathrm{d}\gamma'' \exp\left\{\frac{\mathrm{i}}{2}(p-q)\alpha'' + \frac{\mathrm{i}}{2}(p+q)\gamma''\right\} Q(\beta'', \beta'; \alpha''; \gamma''; \sigma) \tag{5.10}$$

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† Note the misprints in [3]. The expression $(\mathrm{i}MR^2/\hbar^2)$ in formulae (3.10) and (3.11) should read $(\mathrm{i}MR^2/\hbar\sigma_j)$.

where

$$Q(\beta'',\beta';\alpha'';\gamma'';\sigma) = \lim_{N\to\infty} \int_{SU(1,1)} \prod_{j=1}^{N} \left(\frac{MR^2}{2\pi i\hbar\sigma_j}\right)^{1/2} \left(\frac{iMR^2}{2\pi\hbar\sigma_j}\right) \exp\left\{\frac{i}{\hbar}S_j\right\}$$

$$\times \prod_{j=1}^{N-1} \frac{1}{8}\sinh\beta_j \, d\beta_j \, d\alpha_j \, d\gamma_j$$

(5.11)

is an $SU(1,1)$ path integral [4] expressed in terms of Euler angles (α,β,γ) with

$$S_j = \frac{MR^2}{\sigma_j}\left[1 - \cosh(\Omega_j/2)\right]$$

(5.12)

and

$$\cosh(\Omega_j/2) = \widehat{\cosh^2(\beta_j/2)}\cos[(\Delta\alpha_j + \Delta\gamma_j)/2] - \widehat{\sinh^2(\beta_j/2)}\cos[(\Delta\alpha_j - \Delta\gamma_j)/2].$$

(5.13)

In this manner we have realised the dynamical $SU(1,1)$ symmetry of the Coulomb problem in the hyperbolic space by changing the radial path integral (4.11) into one over the $SU(1,1)$ dynamical group manifold.

6. Performing the path integral for the Coulomb problem

The $SU(1,1)$ path integral (5.11) has been evaluated and expressed in terms of $SU(1,1)$ group characters [4, 5]:

$$Q(\beta'',\beta';\alpha'';\gamma'';\sigma) = \frac{1}{2\pi^2}\sideset{}{^{\circlearrowleft}}\sum_{J} d_J \exp\left\{-(i\sigma/\hbar)C_J\right\}\chi^{(J)}(g''g'^{-1}).$$

(6.1)

Here $\sideset{}{^{\circlearrowleft}}\sum_J$ stands for the orthogonal sum of non-equivalent unitary irreducible representations of $SU(1,1)$ of the fundamental series. For the discrete series, which is labelled by half-integers $J = 0, 1/2, 1,\ldots$, the 'dimension' of the representation is $d_J = 2J+1$ and the constant $C_J = (\hbar^2/2MR^2)[(2J+1)^2 - 1/4]$. Choosing a basis which diagonalises the compact generator of $SU(1,1)$ the character in (6.1) can be written as $(\alpha' = \beta' = 0)$:

$$\chi^{(J)}(g''g'^{-1}) = \sum_{\mu,\nu} e^{-i\mu\alpha''}e^{-i\nu\gamma''}V_{\mu,\nu}^J(\beta'')V_{\mu,\nu}^{J\,*}(\beta').$$

(6.2)

Here the Bargmann functions $V_{\mu\nu}^J(\beta)$ play much the same role as the Wigner polynomials in the representation theory of $SU(2)$ and can be expressed by a hypergeometric function (see for example [4]).

Performing the α'' and γ'' integration in (5.10), we obtain the radial promotor,

$$P_l(r'',r';\tau) = (-1/2R^3)(\sinh\beta'\sinh\beta'')^2 \exp\left\{\frac{i\hbar\sigma}{2MR^2}[(2l+1)^2 - (2J+1)^2]\right\}$$

$$\sideset{}{^{\circlearrowleft}}{\sum_J}' d_J \, V_{(p-q)/2,(p+q)/2}^J(\beta'')V_{(p-q)/2,(p+q)/2}^{J\,*}(\beta')$$

(6.3)

where $\sum_{J}'$ is now a restricted sum over representations which allow the values $\mu = (p - q)/2$ and $\nu = (p + q)/2$ for the basis states. In the discrete series this means $\sum_{J=0 \text{ or } 1/2}^{J_0}$, where $J_0 + 1 = (p - q)/2$ and J is either an integer or half-integer depending on whether $(p - q)$ is even or odd, respectively.

With $(\mathrm{d}\tau/\mathrm{d}\sigma) = -4/(\sinh \beta'' \sinh \beta')$ the energy-dependent Green function can be obtained from (6.3) using (4.2):

$$G_l(r'', r'; E) = \frac{2\pi M}{\mathrm{i}\hbar^2 R} \sinh \beta' \sinh \beta'' \sum_{n_r=0}^{J_0} \delta(J_0 - n_r - l) V_{J_0+1,(p+q)/2}^l(\beta'') V_{J_0+1,(p+q)/2}^{l\,*}(\beta').$$

$$(6.4)$$

As in the case of the Coulomb problem in spherical space [3], only the discrete series with $J = l$ contributes to (6.4). Note that we have changed the sum over J into one over $n_r = J_0 - J = 0, 1, 2, \ldots \leq J_0$.

The poles of (6.4) determine the energy spectrum. Since $J_0 = n_r + l$, we find $(p - q)/2 = n$, where $n = n_r + l + 1$. With (5.8) the resulting spectrum is

$$E_n = -\frac{\hbar^2}{2MR^2}(n^2 - 1) + \frac{Ze^2}{R} - \frac{MZ^2e^4}{2\hbar^2 n^2} \qquad (6.5)$$

where $n = 1, 2, 3, \ldots$. This is identical with that obtained by Infeld and Hull [12] using the factorisation method of Schrödinger [13].

The energy eigenfunctions corresponding to (6.5) can be found by calculating the radial propagator (see (4.3)):

$$K_l(r'', r'; t'' - t') = \sum_{n=l+1} \exp\left\{-\frac{\mathrm{i}}{\hbar} E_n(t'' - t')\right\} R_{nl}(r'') R_{nl}^*(r') \qquad (6.6)$$

where

$$R_{nl}(r) = \left[\frac{\varepsilon_n^2 - n^2}{R^3 n}\right]^{1/2} \sinh \beta \; V_{n,\varepsilon_n}^l(\beta) \qquad (6.7)$$

are the normalised wavefunctions and $\varepsilon_n = MRZe^2/\hbar^2 n$. Note that the energy spectrum (6.5) and normalised energy eigenfunctions (6.7) for the Coulomb problem in hyperbolic space can also be obtained from the one in spherical space [3] by analytic continuation in $R \to \mathrm{i}R$. The Bargmann function in (6.7) is explicitly given by

$$V_{n,\varepsilon_n}^l(\beta) = \frac{(-1)^{n+n_r-\varepsilon_n}}{\Gamma(2l+2)} \left(\frac{\Gamma(1+l+\varepsilon_n)\Gamma(1+l+n)}{\Gamma(\varepsilon_n - l)\Gamma(n - l)}\right)^{1/2} [\cosh(\beta/2)]^{-n-\varepsilon_n}$$

$$\times [\sinh(\beta/2)]^{n+\varepsilon_n-2l-2} {}_2F_1(1 - n + l, l - \varepsilon_n + 1; 2l + 2; -\sinh^{-2}(\beta/2)). \quad (6.8)$$

Using the relations

$$\sinh \chi = \frac{1}{\sinh \beta} \qquad \sinh^2(\beta/2) = \frac{\mathrm{e}^{-\chi}}{2 \sinh \chi} \qquad \cosh^2(\beta/2) = \frac{\mathrm{e}^{\chi}}{2 \sinh \chi}$$

we can transform the radial wavefunction (6.7) back to the χ variable:

$$R_{nl}(r) = \frac{2^{l+1}}{\Gamma(2l+2)} \left(\frac{(\varepsilon_n^2 - n^2)}{R^3 n} \frac{\Gamma(1 + \varepsilon_n + l)\Gamma(1 + n + l)}{\Gamma(\varepsilon_n - l)\Gamma(n - l)}\right)^{1/2} \sinh^l \chi$$

$$\times \exp\{-\chi(n + \varepsilon_n - l - 1)\} {}_2F_1(1 - n + l, 1 - \varepsilon_n + l; 2l + 2; 1 - \mathrm{e}^{-2\chi}). \quad (6.9)$$

Here we have ignored an unimportant constant phase factor appearing in (6.8).

In the flat-space limit $R \to \infty$ the energy spectrum (6.5) goes over to the standard formula, $E_n = -MZ^2e^4/2\hbar^2n^2$. Similarly, using the following limiting relations with $a = \hbar^2/MZe^2$ and $\varepsilon_n = R/an$:

$$\lim_{R \to \infty} {}_2F_1(1 - n + l, 1 - \varepsilon_n + l; 2l + 2; 1 - e^{-2\chi}) = {}_1F_1(1 - n + l; 2l + 2; 2r/an)$$

$$\lim_{R \to \infty} \exp\{-\chi(n + \varepsilon_n - l - 1)\} = \exp\{-r/an\}$$

$$\lim_{R \to \infty} \sinh^l \chi \left[\frac{(\varepsilon_n^2 - n^2)}{R^3n} \frac{\Gamma(1 + l + \varepsilon_n)}{\Gamma(\varepsilon_n - l)}\right]^{1/2} = (r/an)^l \left[\frac{1}{n^4a^3}\right]^{1/2} \tag{6.10}$$

we obtain the well known flat-space wavefunctions:

$$R_{nl}(r) = \left[\left(\frac{2}{na}\right)^3 \frac{(n + l)!}{2n(n - l - 1)!}\right]^{1/2} \frac{(2r/na)^l}{(2l + 1)!} e^{-r/na} \, {}_1F_1(1 - n + l; 2l + 2; 2r/na). \tag{6.11}$$

7. Concluding remarks

In the present work we have explicitly performed the path integral for the hydrogen atom in hyperbolic space. This problem has only briefly been discussed by Infeld and Hull [12] using the factorisation method. However, a general local spacetime transformation in path integrals,

$$x_j = f(y_j) \qquad \tau_j = f'(y_{j-1})f'(y_j)\sigma_j \tag{7.1}$$

corresponds to the following change in the Schrödinger equation:

$$x = f(y) \qquad \psi(x) = (f'(y))^{1/2} \varphi(y). \tag{7.2}$$

Therefore, the radial Schrödinger equation for the Coulomb problem in hyperbolic space may easily be transformed into the differential equation of the hypergeometric functions by making the replacements, $\beta = -\log\tanh(\chi/2)$ and $\psi(\chi) = (\sinh\beta)^{1/2} \varphi(\beta)$.

On the other hand, as we have shown in this paper, the system under consideration has a dynamical $SU(1, 1)$ symmetry. Hence, the algebraisation of this problem may also be easily performed. Such a treatment should be very much similar to that of the Kepler problem in spherical space [14] (see also [6]).

There is also another point worth mentioning. In the case of the hydrogen atom in spherical space [3], the entire spectrum is discrete. In the flat-space limit the large values of n, comparable with R, such that $n = kR$ (k = constant), degenerate into the continuous spectrum $E = \hbar^2k^2/2M$. In the present case, we have considered only the bound states belonging to the discrete spectrum. Similar to the classical condition $\lambda < 1/2$, where $\lambda = L^2/(2MRZe^2)$, the angular momentum for bound states has the upper limit (4.16). Furthermore, with the requirement $E_n > V_{\text{eff}}^q(\chi_{\min})$ one obtains the condition $n^2l(l + 1) < 1/(2\Lambda)^2$. Together with (4.16) this leads to an upper limit for the principal quantum number n:

$$n < 1/\sqrt{2\Lambda} \tag{7.3}$$

which is identical to that of Infeld and Hull [12]. Note that the parameter Λ describes the relation between the Bohr radius $a = \hbar^2/(MZe^2)$ and the radius of curvature R, that is, $\Lambda = a/2R$.

In the spherical case, we have observed previously [3] that only states available in the Kepler problem are those corresponding to the discrete spectrum. In the limit of vanishing curvature, part of the discrete spectrum degenerates so as to form a continuous spectrum extending from zero to infinity. In the hyperbolic case, in addition to the discrete states, there are scattering states whose energies form a continuous spectrum ranging from $\hbar^2/(2MR^2)$, which is the contribution of the curvature, to infinity. Unlike the spherical case, no part of the discrete spectrum of the hyperbolic system contributes to the continuous spectrum in the flat-space limit. The upper bound of the angular momentum tends to infinity as the curvature vanishes, so that the discrete spectrum of the hyperbolic system generates the entire discrete spectrum of the usual flat-space hydrogen atom upon vanishing of curvature.

Acknowledgments

References

[1]   Gutzwiller M C 1985 *Phys. Scr.* **9** *184*
      Balazs N C and Voros A 1986 *Phys. Rep.* **143** 109
      Aurich R, Sieber M and Steiner F 1988 *Phys. Rev. Lett.* **61** 483
[2]   Gutzwiller M C 1980 *Phys. Rev. Lett.* **45** 150
[3]   BarutA O, Inomata A and Junker G 1987 *J. Phys. A: Math. Gen.* **20** 6271
[4]   Junker G and Böhm M 1986 *Phys. Lett.* **117** 375
[5]   Böhm M and Junker G 1987 *J. Math. Phys.* **28** 1978
[6]   Barut A O, Inomata A and Wilson R 1987 *J. Phys. A: Math. Gen.* **20** 4075, 4083
[7]   Inomata A and Kayed M A 1985 *J. Phys. A: Math. Gen.* **18** 235; *Phys. Lett.* **108A** 9
[8]   Inomata A 1988 *Path Summation: Achievements and Goals* ed S Lundqvist, A Ranfagni, V Sa-yakanit and L S Schulman (Singapore: World Scientific) p 114
[9]   Barut A O and Raczka R 1980 *Theory of Group Representations and Applications* (Warsaw: Polish Scientific Publishers) p 444
[10] Böhm M and Junker G 1988 *Path Summation: Achievements and Goals* ed S Lundqvist, A Ranfagni, V Sa-yakanit and L S Schulman (Singapore: World Scientific) p 469
      Junker G 1989 *Path Integrals from meV to MeV* ed V Sa-yakanit *et al* (Singapore: World Scientific) p 127
[11] Peak D and Inomata A 1969 *J. Math. Phys.* **10** 1422
[12] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
      Infeld L and Schild A 1945 *Phys. Rev.* **67** 121
[13] Schrödinger E 1940 *Proc. R. Irish Acad.* A **46** 9; 1941 *Proc. R. Irish Acad.* A **46** 183; *Proc. R. Irish Acad.* A **A47** 53
[14] Barut A O and Wilson R 1985 *Phys. Lett.* **110A** 351